ASSERTING FAIRNESS THROUGH AI, MATHEMATICS AND EXPERIMENTAL ECONOMICS

The CREA Project Case Study

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Abstract. This is an account of Analytical-Experimental Workgroup role in a two-year EU funded project. A restricted group of economists and mathematicians has interacted with law researchers and computer scientist (in the proposal's words) "to introduce new mechanisms of dispute resolution as a helping tool in legal procedures for lawyers, mediators and judges, with the objective to reach an agreement between the parties". The novelty of the analysis is to allow different skills (by legal, experimental, mathematical and computer scientists) work together in order to find a reliable and quick methodology to solve conflict in bargaining through equitable algorithms. The variety of specializations has been the main challenge and, finally, the project's strength.

Keywords: *Algorithms, Private law, Game Theory, Social Choice, Fair Division*.

1. Introduction

Law school programs typically do not provide any mathematic training. Symmetrically, STEM students do not have any exposure to the law principles. Moreover, in real life fair division problems involving both capabilities are quite frequent and imply relevant personal and social costs.

The path that connects a topic known as fair division theory to the definition of AI algorithms in the realm of private law is a long and

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winding one. The problem of fair division[5] originates as a set of recreational problems that Hugo Steinhaus, a prominent Polish mathematician, discussed with his students Stefan Banach and Stanislaw Ulam (who later became just as important as their teacher) around a table of the Scottish Café in Lvov (then in Poland, now Lviv in Ukraine) during WW2[6]. For many decades that followed the topic remained a source of inspiration for a few enlightened mathematicians and economists that published their finding in highly reputable journals. By the end of the millennium, the number of articles and books that covered fair division increased, and fair division became a regular topic in worldwide conferences of social choice theory, a theoretical framework for analysing and combining individual opinions, preferences, interests to reach a collective decision. Fair division was now applied to many different areas, from sewage treatment (Goetz, 2000)[7] to gerrymandering (Hill, 2000)[8], and detailed examples of divorces too were now being explained with the tools of fair division the first notable example being that of (Brams and Taylor, 1996)[9].

With the new millennium, two other trends emerged: On the one side, computer scientists turned their attention to the economic sciences at large. Fair division, dealing the allocation of physical, as well as virtual resources, became a natural outlet for their studies. Starting from Lipton et al. (2004)[10], the design of algorithms, as well as the analysis of their computational complexity, became the focal point of what is now known as computational fair division.

The last fifteen years also witnessed a rising interest by experimental economists towards fair division issues. Those researchers test the validity of theoretical findings by designing laboratory experi-

[5] For a general introduction to the topic we refer to Brams, S.J., Taylor, A.D. (1996): Op. Cit. and Moulin, H., (2003). Fair Division and Collective Welfare. MIT Press.

[6] For an account of their activity, we refer to Hill, T.P. (2000). Slicing Sandwiches, States, and Solar Systems. The American Scientist, Vol.106 (1), 54-61.

[7] Goetz, A. (2000). Cost Allocation: An Application of Fair Division, The Mathematics Teacher, 93 (7), 600--603.

[8] Op.Cit.

[9] Brams, S.J., Taylor, A.D. (1996): Fair-Division – From cake-cutting to dispute resolution, Cambridge University Press New York (USA).

[10] Lipton, R.J. Markakis, E., Mossel, E. and Saberi, A. (2004). On approximately fair allocations of indivisible goods . In Proceedings of the 5th ACM Conference on Electronic Commerce (EC), 125--131.

ments in which real agents take their decision in neatly defined procedures that make the theoretical frameworks real. One of the first work that explicitly aims at validating a fair division procedure, namely the Adjusted Winner, is provided by Daniel and Parco (2005)[11].

The first applications of fair division procedures within the legal context, can now be newly examined with updated paraphernalia that includes a greater computational capability and specifically designed testing tools.

We note, however, that the driving force that motivates every work is one that originates from theoretical findings, with the potential end users relegated to the background. We aimed to change the perspective and designed a project that could bring together designers and end users to work side-by-side in a joint effort.

In this note we refer about how the task of finding an equitable and feasible innovative methodology has been performed by the Analytical-Experimental Workgroup of the European CREA Project[12] through a chronological description of the activities of a restricted group of economists and mathematicians within the project.

The challenge of the project has been to involve researchers with very different background to design an algorithmic procedure which is suited for the family law context, that guarantees fairness and that is perceived as "fair" by the users. This task has been achieved by making extensive use of the computational and experimental tools that have been developed recently. Two competitive models of fair division are here exemplified, coded and tested experimentally.

2. The State of the Art in Fair Division Procedures.

[11] Daniel, T.E. and Parco, J.E. (2005), Fair, Efficient and Envy-Free Bargaining: An Experimental Test of the Brams-Taylor Adjusted Winner Mechanism, Group Decision and Negotiation, 14, 241-264.

[12] "Conflict Resolution through Equitable Algorithms" – CREA in short – project, financed by the Justice Programme of the European Union (Grant Agreement No. 766463, Call: JUST-AG-2016-05) and led by Prof. Francesco Romeo (University "Federico II" of Naples)

The first three months of the workgroup activity have been devoted to an updated review[13] of the existing procedures in fair division theory. In the review, the most recent results have been examined, restricting the focus on a smaller area of research characterized by the following indications:

- Results regarding the problem of allocating several objects (also referred to as items, goods) to a finite number of persons, usually referred to as agents or players.
- A focus for ready-made procedures that could be, in principle, straightforwardly adapted to the legal context at hand.
- Mathematical notation has been avoided whenever possible, with the belief that the lack of mathematical precision can be more than compensated by the opportunity to reach a larger audience.

First of all, a review of the methods to represent the agents' preferences has been analysed. Preferences can be cardinal or ordinal. The former indicates that a number, usually called utility, describes the satisfaction induced by the reception of the item by a particular agent. When the latter is used, agents only indicate whether they prefer one allocation to another. When the agents elicit their preferences, they may consider a simpler method and indicate their preferences only for the single items – thus assuming that the benefit that each item induces when it is assigned to an agent, is independent from that of the other goods. Alternatively, the agents should record the benefit that each combination (bundle) of goods induces to each agent. The latter method is essential when items are of similar nature or they are closely coupled, but the whole elicitation process may require a huge effort on the agents' side.

Procedures can be sorted according to the preference method used, on whether goods must be assigned in their entirety to one agent only, or they can be split among two or more agents. Sometimes items can be physically split (think for instance about money, or a piece of land). When an actual split is not possible, the good's ownership may

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[13] The review has been the subject of the first deliverable of the workgroup. Its content has been edited and reworked in Dall'Aglio, M., Di Cagno, D. and Fragnelli, V., Fair Division Algorithms and Experiments: A Short Review, in Algorithmic Conflict Resolution – Fair and Equitable Algorithms in Private Law (Romeo, F., Dall'Aglio, M. and Giacalone, M. eds), Giappichelli, 2019, 155-184.

be divided, or one agent may buy the ownership off the others – becoming the only possessor of the good. When ordinal preferences are used, only indivisible goods can be considered, because it is not possible to evaluate fractions of goods.

In the review of the existing methods, it emerges that the case with two agents has been widely explored, but most of these methods cannot be extended to the case of three or more agents.

We now give a list of the examined procedures, dividing them in 5 groups.

- **Two agents – ordinal preferences -- indivisible goods**
    - The Undercut Procedure (Brams, Kilgour and Klamler, 2012)[14]
    - The Trump rule (Pruhs and Woeginger, 2012)[15]
    - The AL Procedure (Brams, Kilgour and Klamler 2014)[16]
    - The Singles-Doubles and Iterated Singles Doubles procedures (Brams. Kilgour and Klamler, 2017b)[17]
- **Any number of agents -- ordinal preferences -- indivisible goods**
    - The SA Procedure (Brams, Kilgour and Klamler, 2017a)[18]
    - Picking sequences (Brams and Taylor, 2000 and Bouveret and Lang, 2011)[19]

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[14] Brams, S.J., Kilgour, D.M., Klamler, C. (2012) The undercut procedure: an algorithm for the envy-free division of indivisible items, Social Choice and Welfare, 39, 615-631.

[15] Pruhs K., Woeginger GJ (2012): Divorcing made easy. In: Kranakis E, Krizanc D, Luccio F (eds.) FUN 2012, LNCS 7288, Springer, Berlin, pp 305-314.

[16] Brams S.J., Kilgour D.M., Klamler C. (2014): Two-Person Fair Division of Indivisible Items: An Efficient, Envy-Free Algorithm". Notices of the American Mathematical Society, 61 (2), pp.130-141.

[17] Brams, S.J., Kilgour, D.M., Klamler, C. (2017b) Maximin Envy-Free Division of Indivisible Items, Group Decision and Negotiation, 26, 115-131.

[18] Brams, S.J., Kilgour, D.M., Klamler, C. (2017a) How to divide things fairly, Mathematics Magazine, 88 (5), pp. 338-348.

[19] Brams, S. J. and Taylor, A. D. (2000): The Win-Win Solution: Guaranteeing Fair Shares to Everybody.New York: W. W. Norton and Bouveret, S. and Lang, J. (2011): A General Elicitation-Free Protocol for Allocating Indivisible Goods.In Proceedings of the 22nd International Joint Conference on Artificial Intelligence

   - The Descending Demand Procedure (Herreiner and Puppe, 2000)[20]

- **Two agents – cardinal preferences – divisible goods**
   - The Adjusted Winner procedure (Brams and Taylor 1996)[21]

- **Any number of agents – cardinal preferences – divisible goods**
   - The Egalitarian solution (Olvera-Lopez and Sanchez-Sanchez, 2014 and Dall'Aglio, Di Luca and Milone, 2017)[22]
   - The Nash Product Maximizer for divisible goods (Bogomolnaia, et al. 2017)[23]

- **Any number of agents – cardinal preferences – indivisible goods**
   - The Envy Cycle Procedure (Lipton et al., 2004)[24]
   - The Nash Product Maximizer for indivisible goods (Caragiannis et al., 2016)[25]

More details about the listed procedures can be found in Dall'Aglio, Di Cagno and Fragnelli (2019)[26]. In addition to the general procedures listed above, a special class of algorithms stands out for its relevance

(IJCAI), 73-78. Palo Alto, CA: AAAI.

[20] Herreiner, D., Puppe, C. (2002): A simple procedure for finding equitable allocations of indivisible goods. Social Choice and Welfare, 19, 415-430.

[21] Op. Cit.

[22] Olvera-López, W., Sánchez-Sánchez, F. (2014): An algorithm based on graphs for solving a fair division problem, Operations Research - An International Journal, 14 (1), 11-27 and Dall'Aglio, M., Di Luca, C. and Milone, L. (2017): Finding the Pareto optimal equitable allocation of homogeneous divisible goods among three players, Operations Research and Decisions, 27(3), 35-50.

[23] Bogomolnaia, A., Moulin, H., Sandomirskiy, F. and Yanovskaya, E. (2017): Competitive division of a mixed manna, Econometrica, 85: 1847-1871.

[24] Lipton, R., Markakis, E., Mossel, E., and Saberi, A. (2004): On Approximately Fair Allocations of Indivisible Goods. In Proceedings of the 5th ACM Conference on Electronic Commerce (EC), 125-131. New York: ACM.

[25] Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia A.D., Shah, N. and Wang J. (2016): The Unreasonable Fairness of Maximum Nash Welfare. In Proceedings of the 2016 ACM Conference on Economics and Computation (EC '16). ACM, New York, NY, USA, 305-322.

[26] Op. Cit.

in Private Law: that of fair division involving money. Here we distinguish between two classes of methods:

- Procedures that consider the allocation of goods and money (Alkan et al., 1991 and Bevia, 1998)[27]. Here money is used to level out the disparities that may arise when indivisible goods are allocated.
- Procedures that consider the fact that goods have a market value. Bellucci and Zeleznikow (2006)[28] defined the Family Winner procedure, later perfected in Bellucci (2008)[29] as the Asset Divider procedure for two agents. Here agents express preferences in the same way as Brams and Taylor (1996)[30] do in the Adjusted Winner procedure on goods that have a market value. Karp, Kazachkov and Procaccia (2014)[31] consider a model for two agents where items can be sold at a fraction of their value, and measure how this option improves the resulting allocation in terms of the agents' satisfaction.

The review has pointed out a consolidated stream of works that illustrate fair division procedures with a solid theoretical background. With very few exceptions, each procedure is based on a well-defined mathematical model, and the proposed allocation satisfies the properties defined in advance. The most recent models, however, lack a serious testing ground and a credible feedback from actual users. A notable exception is given by two web resources: The Adjusted Winner website[32] and by the Spliddit portal[33]. The procedures are well ex-

[27] Alkan, A., Demange, G., Gale D. (1991): Fair allocation of indivisible goods and criteria of justice, Econometrica, 59 (4), 1023-1039 and Bevia, C. (1998): Fair allocation in a general model with indivisible goods, Review of Economic Design, 3, 195-213.

[28] Bellucci, E., and Zeleznikow, J., (2006). Developing negotiation decision support systems that support mediators:a case study of the Family Winner system. Artificial Intelligence and Law, 13(2), 233-271.

[29] Bellucci, E., (2008). AssetDivider: A New Mediation tool in Australian Family Law. In Hindriks, K.V., & Brinkman, P-W., (Eds.), HUCOM 2008 -1st InternationalWorking Conference on Human Factors and Computational Models in Negotiation (pp. 11-18). Association for Computing Machinery.

[30] Op. Cit.

[31] Karp, J.A., Kazachkov, A.M., and Procaccia, A.D., (2014). Envy-free division of sellable goods. The Twenty-Eighth AAAI Conference on Artificial Intelligence, 728-734.

[32] The Adjusted Winner website: https://pages.nyu.edu/adjustedwinner/

[33] The Spliddit portal: http://www. spliddit. org

plained in plain terms and references for a better insight are given. Moreover, the websites managers can be contacted for a feedback.

From an experimental point of view there exists a huge literature testing fairness/inequity aversion testing in the lab (see Camerer, 2011[34] for reference) with divisible goods checking also for participants heterogeneity. Also, indivisible good experiments show a great concern of participants on inequality aversion. More in line with our research question Bouveret and Lemaitre (2016)[35] investigate five different fairness criteria in the lab and Schneider and Kramer (2004)[36] developed an experimental comparison of three different procedures of fair division in the lab.

2.1. Two general criteria: The Egalitarian and the Competitive/Nash solutions

How to find an optimal allocation of the goods? Research suggests that no single criterion is universally better than others. The recent literature shows that two criteria prevail:

2.1.1. The Egalitarian allocation

If the agents have the same importance, measured by their entitlement, this solution guarantees that the agents achieve the same satisfaction level and this level is as high as possible. In case of different entitlement quotas, equality is attained once values are weighted with the shares in order to attain equality. This solution was introduced by Pazner and Schmeidler (1978)[37] and it is equivalent to the egalitarian

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[34] Camerer, C.F., (2011). Behavioral Game Theory: Experiments in Strategic Interaction, Princeton University Press.

[35] Bouveret, S., Lemaître, M. (2014): Characterizing conflicts in fair division of indivisible goods using a scale of criteria. In: Proceedings of the 13th International Conference on Autonomous Agents and Multiagent Systems, pp. 1321–1328. IFAAMAS.

[36] Schneider, G. and Kramer, U.S. (2004), The Limitations of Fair Division: An experimental evaluation of three procedures, Journal of Conflict Resolution, 48 (4), 506-524.

[37] Pazner E and Schmeidler D. (1978): Egalitarian equivalent allocations: A new concept of economic equity, Quarterly Journal of Economics, 92 (4), 671-687.

solution in bargaining problems proposed by Kalai and Smorodinsky (1975)[38] (see also Kalai, 1977)[39].

By construction, this solution is egalitarian. i.e. it provides the same utility vs. entitlement ratio level for all the agents. It turns out that this solution is also **Pareto optimal or Efficient**[40]. The allocation, however, fails to verify other important properties. For instance, the allocation may cause one or more players to be envious of the goods assigned to other players.

2.1.2. The Competitive/Nash Allocation.

Instead of simply allocating the goods, we may figure out a market situation in which each agent has a budget given by the player's share of the total market value of the disputed goods and buys goods sold at commonly known prices. If prices are fixed in a way that

a) each player, independently of the others, makes the best choice given the budget, he/she buys goods that maximize his/her own satisfaction and

b) all goods are sold with no overlaps (for instance two agents buying the same good in its entirety) and no leftovers (no good remains unsold),

the goods' distribution is called the Competitive Equilibrium from Equal Income (CEEI) solution, while prices are indicated as equilibrium prices. This Competitive Equilibrium allocation coincides with the solution proposed by John Nash (Nash 1950)[41] for bargaining problems in which the sum of the logarithms of the players' utility is maximized. This solution is usually refer

[38] Kalai E, Smorodinsky M (1975): Other solutions to Nash's bargaining problem. Econometrica. 43, 1975, 513-518.

[39] Kalai E. (1977): Proportional solutions to bargaining situations: Intertemporal utility comparisons. Econometrica. 45, 1623-1630.

[40] Informally, an allocation is Pareto-efficient if it cannot be improved to another allocation which is at least as good for every agent and strictly better for at least one agent.

[41] Nash J.F. (1950): The bargaining problem. Econometrica 18, 155-162.

This solution, usually referred to as the Competitive/Nash solution. is[42] **Pareto optimal or Efficient**. The solution is also **Envy-Free**[43]**.** However, the solution is usually not egalitarian, since the outcome may yield different utility levels for the agents involved. This may raise some questions if the evaluations concern money.

3. Real Data for the Theoretical Models

As next step, the workgroup activities have been devoted to an analysis of actual court cases where a number of assets need to be divided among two or more parties. The review has been the subject of the second deliverable of the workgroup which has been edited and reworked in Dall'Aglio (2019a)[44].

The project has experienced a fruitful interaction with the law researcher who formed the Legal Workgroup within the same project. The team had worked in the previous months to describe 36 actual cases in the fields of Family Law (succession and divorce) and Company Law (liquidation) that, in the researchers' view, might benefit from an algorithmic treatment. After the description of each case, we provided a comment, with a description of how the existing procedures could yield a solution, but also an analysis of the new features to implement on a new product. As a result of this work, a list of the improvements for a general-purpose procedure was compiled. The ideal procedure should include the following features:

a. It should avoid random outcomes.

b. It should be able to deal with agents having different shares of entitlement.

[42] The following two informal definitions are taken from Bouveret S., Chevaleyre Y., Maudet N. (2016): Fair allocation of indivisible goods, Chapter 12 of Handbook of Computational Social Choice (Felix Brandt, Vincent Conitzer, Ulle Endriss, Jerome Lang, Ariel D. Procaccia, eds), Cambridge University Press, New York.

[43] Informally, an allocation is envy-free if no agent prefers the share of another agent to her own.

[44] Dall'Aglio, M., (2019a): Fair Division and the Law: How Real Cases Helped Shape Allocation Procedures in the Legal Setting across European Countries, in Algorithmic Conflict Resolution - Fair and Equitable Algorithms in Private Law (Romeo, F., Dall'Aglio, M. and Giacalone, M. eds), Giappichelli, 2019, 185-222.

c. It should be able to consider allocations where items and/or money are preventively assigned to one of the agents.

d. It should take into account that certain items may have to remain indivisible, and, therefore, these have to be assigned to one of the agents in their entirety.

e. It should consider the need imposed by the Law or by the circumstances to certain agents for liquid assets.

f. It should encompass liabilities, as well as assets.

In some situations, further restrictions may be imposed on the division. Here we list some instances.

- Indivisible goods. The law or the court may require that an item is assigned in its entirety to one of the agents.

- Assignment restrictions. Not all the conceivable assignments may be acceptable as solutions because they may infringe some requirement of the law or some ruling of the court. The procedures must rule out such inadmissible results. We consider two notable classes of restricted admissibility:

    - Simple assignment restrictions. One or more goods may be assigned to a specific agent or one agent in a restricted group.

    - Joint assignment restrictions. Two or more goods may not be assigned to the same agent. More in general, a combination of assignments among agents may be inadmissible.

We notice that these are additional restrictions imposed on the solution that may impoverish the solution. For instance, imposing goods to be assigned in their entirety to one of the agents may generate envy among the agents.

In many cases the algorithms can be applied straightforwardly, but there are issues that should be taken care of by more refined versions of the algorithms. For instance, a preliminary routine should compute how the entitlements of the involved agents change as resulting from the contribution or dissipation that each agent has brought to the asset in the period where such asset was managed in common by the agents who are now participating to its division.

More delicate issues, such as child custody in a divorce, should be handled with great care, since they cannot be treated as any other item in the contended asset. We therefore recommend the exclusion of such issues from an algorithmic treatment.

4. Game Theory at Work

The real-life cases described by the Legal Workgroup of the project highlighted the need for new procedures based on strong theoretical grounds, but flexible enough to handle a diversity of situations. This was the starting point of a new project phase.

4.1. Adapting the general principles.

Most of the existing fair division procedures cannot be applied straightforwardly because in the most common mathematical models of utility it is not possible to merge the agents' preferences together with the goods' market value. The two works that explicitly deal with the allocation of goods with market value are the already mentioned works by Bellucci (2008)[45] and Karp et al. (2014)[46]. While both works deal with the two-agent case, the former defines a procedure whose goal is, in the author's words, "to provide feasible suggested solutions to the conflict that are acceptable to the user, which for our purposes does not involve searching for optimal solutions as in Pareto optimisation." Therefore, a proper optimization set up is not considered by that author. Karp et al. (2014)[47] considers the allocation of indivisible goods between two agents, introducing the option to sell a good and divide the resulting cash to make the division fairer. The goal of this paper is not to set up new procedure, but to show how the division improves when the selling option is introduced.

4.2. Keeping manipulability under control

In Dall'Aglio and Fragnelli (2019)[48], a comparison of the robustness against manipulation from one of the agents has been carried out.

[45] Op. Cit.
[46] Op. Cit.
[47] Op. Cit.

Even in a simplified setting that consider two agents competing over two divisible goods, with each of the agents completely informed over the real preferences of the other, it turns out that none of the procedures is entirely immune from strategic fiddling of the informed agent's bid in order to get a larger share at the expenses of the other participant. The two procedures, however, behave differently with the Competitive/Nash solution outperforming the egalitarian one. In fact, it was proved that:

- The Competitive/Nash solution does not change at all for small alterations of the informed agent's bid. A change occurs only for larger deviations. In particular, a gain occurs only when the informed agent declares with his bids that his most preferred item is actually the least preferred.
- The gain that the informed agent can get from manipulating his preferences is always at least as great with the Egalitarian solution than with the Competitive/Nash one. This happens, in particular, at the maximum gain level that the informed agent can achieve.

4.3. Towards the procedures

Once the preliminary analysis had been carried through, the team was ready to set up new procedures. A detailed description of the work is given in Dall'Aglio (2019b)[49].

When agents have preferences over goods with market values, it would be natural to set up a problem in which an optimal allocation is found that yields bundles of goods of exact equal market values (or, more generally, proportional to the agents' entitlement. It has been shown through a series of counterexamples, that such goal may come at a cost of having too many split items in the recommended allocation. This difficulty has been overcome by an accurate utility modelling and by setting up proper mathematical goals which allow for differences in the market value among the agents but give proper justi-

[48] Dall'Aglio, M. and Fragnelli, V., (2019): On the Manipulability of the Division of Two Items Among Two Agents, in Algorithmic Conflict Resolution - Fair and Equitable Algorithms in Private Law (Romeo, F., Dall'Aglio, M. and Giacalone, M. eds), Giappichelli, 2019, 223-230.

[49] Dall'Aglio, M., (2019b): Fair Division Procedures for the CREA project, in Algorithmic Conflict Resolution - Fair and Equitable Algorithms in Private Law (Romeo, F., Dall'Aglio, M. and Giacalone, M. eds), Giappichelli, 2019, 231-272.

fication for such differences. Two utility models have been matched with two optimization goals. In both cases the goods' market values is common knowledge among the agents.

4.4. Utility as bids with the Nash solution

In a first proposal, agents express their preferences as bids over the goods. Clearly, the higher the bid, the more palatable is a given good for an agent. Agents do not bid their own money, but rather use a virtual budget which is given by the total market value of the contended goods. Also, the agents' bid cannot be too low because goods have an intrinsic market value. The idea is to fix a lower bound, say 25% lower than the market value. Since a higher bid on a good signals a greater interest on that item, we defined the good's utility for a given agent as equal to the bid. In this context, it is natural to consider the Competitive/Nash solution for two reasons.

- It provides an envy-free solution.
- It defines an equilibrium price for each good. These prices usually differ from the market values, but they fully justify the resulting allocation.

4.5. Utility as rating with the Egalitarian solution.

In this model, agents are simply asked to express their preference over the goods by means of a limited number of levels, typically 5, as it happens when users rate a good on Amazon or a restaurant on Trip Advisor. The rating is used to modify a good's utility, magnifying the market value if the rating is high or, conversely, by shrinking it if the rating is low. In this context, the Egalitarian solution was chosen, with the following explanation in mind: in case of equal entitlements, agents will receive bundles that do not have the same market value, but the differences can be explained in terms of the quality of the received goods, i.e., the different average rating that each agent assigns to those goods.

5. The procedures.

We provide a detailed description of the two different procedures designed by one of the authors within the context of the CREA project. Those descriptions are taken from Section 6 in Dall'Aglio (2019b)[50].

5.1. Procedure 1: Name your price.

In the following, we describe in detail the first procedure. Here, the utility of each item is given by the subjective bids of the agents and the Competitive/Nash solution is sought.

I. PRELIMINARY PHASE. The mediator (or the agents, jointly) insert the following information:

   i. The number and names of the agents.

   ii. The share of entitlement for each agent.

   iii. The number and names of the goods.

   iv. Whether money is one of the assets and whether it should be considered:

      a. As one of the many divisible items to be assigned to one or more agents.

      b. As a "special" resource that could be given to the agents in parts proportional to the shares of entitlement and could help solve questions of joint ownership for the resulting division.

   v. The mediator should determine The sum of the values of the assets. This will be the budget that the agents will spend on the bids.

   vi. For each good, a range of admissible bids should be specified. If the mediator has some idea of the good's market value, the interval should be built around this market value, the lower bound, resp. upper bound,

[50] Op. Cit.

should be determined by subtracting, resp. adding, a fixed percentage, say of the estimated market value.

vii. Whether there exist any constraints on the item

a. an item must be considered indivisible;

b. simple assignment restrictions must be enforced joint assignment restrictions must be enforced.

II. BIDDING PHASE. Each agent is asked to make a bid on every item within a range, if specified in advance. Moreover, the total amount of these bids must not exceed the budget determined by the mediator. Agents must submit their bids independently of each other. To this aim, it is necessary that agents access the web portal in separate sessions.

III. THE SOLUTION. The Competitive/Nash solution, i.e., a solution that maximizes the weighted (by the shares) product of the utilities is sought.

i. In case no restriction has been imposed in step I-vii, the system should present the optimal solution, or one among the optimal solutions, and should explain the optimality properties of the solution:

a. The solution is proportional, i.e. each agent receives a bundle with normalized utility higher than the agent due share.

b. The solution is efficient (or Pareto-optimal), namely no allocation that globally improves the welfare of every agent, with at least one agent strictly better off, is possible.

c. The solution is envy-free, namely any agent prefers the received goods to those given to the others.

ii. For each good, the system will compute an Equilibrium price. The system should then explain that the received goods (or parts thereof) by each agent coincide with the optimal purchase with a virtual budget of goods sold at their equilibrium prices.

iii. In case restrictions have been imposed in step I-vii, the system is not able to compute the equilibrium prices and it should verify whether the computed optimal solution, satisfies any of the properties listed in step i.

a. In the affirmative case, the system should explain the optimality properties shared by the solution.

b. In the negative case, the system should quantify by how much a given property fails to be verify.

5.2. Procedure 2: Price and rate.

The second procedure is less demanding for the agents who only have to provide a rating for each good.

I. PRELIMINARY PHASE. The mediator (or the agents, jointly) insert the following information:

i. The number and names of the agents.

ii. The share of entitlement for each agent.

iii. The number and names of the goods.

iv. The market value of each good.

v. The rating scale for evaluating each item: 3-stars, 5-stars or 7-stars.

vi. Whether money is one of the assets and whether it should be considered:

a. As one of the many divisible items to be assigned to one or more agents;

b. As a “special” resource that could be given to the agents in parts proportional to the shares of entitlement, and could help solve questions of joint ownership for the resulting division.

vii. Whether there exist any constraints on the item:

a. An item must be considered indivisible;

b. Simple assignment restrictions must be enforced;

c. Joint assignment restrictions must be enforced.

II. RATING PHASE. Each agent is asked to indicate the degree of pleasantness for each good according to the rating scale fixed in advance. Each agent proceeds independently of each other,

III. THE SOLUTION. The Egalitarian solution, i.e., a solution that maximizes the normalized utility of the worst-off agent (weighted by its entitlement) is computed.

i. In case no restriction in step I-vii has been imposed, the system should explain the optimality properties of the solution:

a. The solution is proportional, i.e. each agent receives a bundle with normalized utility higher than the agent due share.

b. The solution is efficient (or Pareto-optimal), namely no allocation that globally improves the welfare of every agent, with at least one agent strictly better off, is possible.

c. The solution is egalitarian, namely all the agents will receive the same amount of normalized utility.

ii. In case restrictions have been imposed in step I-vii, the system is not able to compute the equilibrium prices and it should verify whether the computed optimal solution, satisfies any of the properties listed in step i.

a. In the affirmative case, the system should explain the optimality properties shared by the solution.

b. In the negative case, the system should quantify by how much a given property fails to be verify.

6. An Example

We are now going to examine an application of both algorithms to an inheritance case. The example is taken from Dall'Aglio (2019a)[51].

> *During his life, X was the owner of a land plot in Zadar with a building and garden (180m2) with three flats: one on the ground floor (90m2, 180,000 Euros), one on the first floor (60m2, 120,000 Euros) and one on the second floor with a wonderful view of the shore and beach (60m2, 130,000 Euros). […] He also owned another land plot in Zagreb with a building with three flats; one on the ground floor (55m2, where his son A had a mechanic's workshop, 77000 Euros not including equipment), one on the first floor (55m2, where X lived, 80,000 Euros) and one on the second floor (45m2, but needs full renovation, 45000 Euros). This second building was not a condominium. After the death of person X he is succeeded by his sons, A, B and C.*
>
> *A is most interested in the ground floor because he operates a mechanic's workshop which is crucial for his livelihood. He would not mind getting another apartment either in Zagreb or in Zadar.*
>
> *B already had a house in Zagreb, so he was interested in the house in Zadar. He wants two flats, the one on the first floor but especially the one on the second floor (this is his mayor priority).*
>
> *C has a tourist agency, and he wants all flats in Zadar.*

6.1. Procedure 1 – Name your price.

[51] Op.Cit.

A lower bound for the items' prices is fixed in order to represent the minimum offer that each heir is allowed to present. The difference between the market price and the lower bound represents the amount of money that each heir is asked to allocate according to his or her preferences. Finally, the Egalitarian and Competitive/Nash algorithms apply in order to fairly divide the items among the heirs.

1. The lower bounds represent the minimum prices that each heir has to respect for the apartments in the inheritance. Fix a lower bound for the bid: 20% less than the market price. Then, the prices of the six apartments are:

	Zadar			Zagreb		
	GF	1	2	GF	1	2
P	180,000	120,000	130,000	77,000	80,000	45,000
LB	144,000	96,000	104,000	61,000	64,000	36,000

Table 1 - Market prices and lower bounds for the Inheritance example (MP=Market Price, LB=Lower Bound).

2. Let the heirs offer the amount of money they believe the most adequate for each item in the patrimony. The maximum that each heir may allocate when expressing his or her preferences is equal to the maximum value of the sum of all the items in the patrimony, that is Euros 630.000 (The exact sum would be 632000, but for simplicity of communication we prefer a rounder number.)

   Note that no offer can be below the minimum prices expressed by the lower bound.

   For example, A may be willing to offer Euros 100.000 for the ground floor in Zagreb, as he claims the apartment is crucial for his livelihood, and equally redistribute the remaining amount among the other apartments. Mr. B may translate his special preference for the 2nd floor of the building in Zadar with an offer 20% higher than the market price and may offer 10% more than the market price for the 1st floor of the same building, he is not interested at all in the building in Zagreb. Similarly, Mr. C may distribute his preferences equally among the apartments in Zadar.

The allocation of the total offer will be as follows.

	Zadar			Zagreb			Sum
	GF	1	2	GF	1	2	
A	170,000	112,000	123,000	100,000	80,000	45,000	630,000
B	181,000	132,000	156,000	61,000	64,000	36,000	630,000
C	200,000	129,000	140,000	61,000	64,000	36,000	630,000

Table 2 A simulation of the bids compatible with the data of the Inheritance example

Here is the Competitive/Nash Allocation applied to the problem.

- A gets all the flats in Zagreb.
- B gets the second floor in Zadar and a 68% share of the first flat in Zadar.
- C gets the ground floor in Zadar and a 32% share of the first flat in Zadar.

The allocation can be described by Table 3

	Zadar			Zagreb		
	GF	1	2	GF	1	2
A				■	■	■
B		68%	■			
C	■	32%				

Table 3 The "Name your price" solution for the inheritance example

As previously explained, the proposed solution is Proportional, Efficient (Pareto optimal) and Envy-Free. The last property can be summarized by the following table, where the different valuations are described in the rows and the allocations are reported in the columns:

		Allocations		
		A	B	C
Valuations	A	**225,000**	191,300	213,700
	B	161,000	**245,800**	223,200

	C	161,000	218,700	**250,300**

Table 4 The solutions satisfies Envy-Freeness

The valuations of agent A are listed in the first row. That agent values the three flats received as the sum of the respective bids: 100000, 80000 and 45000, yielding 225000 Euros. Agent A values the bundles given to B (The second floor and 68% of the first floor in the Zadar) and to C (The ground floor and 32% of the first floor in Zadar), 191300Euros and 213000Euros, respectively. Agent A has no reason to envy agent B or agent C. A similar reasoning applies to the Agents B and C whose valuation of the received bundle (bold in the table) exceeds that of the bundles assigned to the other agents.

We remark that bids are personal and do not represent objective evaluations. For this reason, a comparison between the values in the main diagonal of Table 4 may induce some agents to complain over having obtained lower values than other agents. In order to avoid any complaint, we recommend that bids are kept private, and each agent has no access to everybody else's valuations.

To explain the solution as an equilibrium, we note that the procedure can compute the following prices for the properties:

	Zadar			Zagreb			Total
	Ground Fl	First Fl.	Second Fl.	Ground Fl	First Fl.	Second Fl.	
Prices	174,000	113,000	133,000	93,500	74,500	42,000	

Table 5 Equilibrium prices for the inheritance case

The equilibrium prices differ from the market values and can be given the following explanation. Suppose that each agent has a budget of 210000 Euros, which corresponds to 1/3 of the total market value of the goods (630000 Euros). Agents are now told that they can buy the houses at their equilibrium prices. Each agent will then optimally spend the budget and will discover that what he/she gets coincides with his/her share in the Competitive/Nash solution. More details are given in Dall'Aglio (2019b)[52].

6.2. Procedure 2 – Price and rate.

[52] Op. Cit.

If the heirs were asked to rate the goods on a 1-to-5 rating scale, a likely outcome would be the following:

	Zadar			Zagreb		
	GF	1	2	GF	1	2
A	*	*	**	*****	***	***
B	***	****	*****	*	*	*
C	*****	*****	*****	*	*	*

Table 6 - A simulation of the ratings compatible with the data of the Inheritance example

The resulting allocation can be described by the following table

		Zadar			Zagreb	
	GF	1	2	GF	1	2
A		2%				
B		64%				
C	7	32%				

Table 7 - The "Name your price" solution for the inheritance example

The resulting allocation is not too different from the one obtained with the first procedure. The only difference is that the first floor flat in Zadar is now split among all three heirs, with agent A bound to receive a tiny fraction of that flat.

The slight difference the two arrangements propose, reflects the different goals that the two solutions pursue: Procedure 1 aims at delivering a solution which is free of envy, since every agent appreciates the received portion of the asset much more than the parts given to the others. Procedure 2 instead cares for delivering parts which do not differ too much in their monetary value, while still caring for the agents' satisfaction.

7. The lab experiment

Through a purpose-built laboratory experiment it is possible to test the procedures adopted.[53]

[53] An account of the Workgroup activity has been the subject of the publicly available fourth deliverable of the workgroup "Deliverable D3.4. Report on Experimental Analysis ", delivered on September 17th, 2019. A detailed description of the work is given in Dall'Aglio, M., Di Cagno, D. and Marazzi F., (2019): Algorithms

The experiment was aimed at eliciting the participants' preference between the two selected procedures, together with their willingness to appeal to an ideal court in case they are not satisfied with the division suggested by the computer. More in details, the experiment consists of a set of different decisional tasks organized into independent phases, that participants play in pairs. At the beginning of the experiments, participants are divided in groups of four so that in each of the three phases they face a different partner, which is then kept constant for all the tasks within the same phase.

The two algorithms proposed in the theoretical part of the project are tested *within subjects,* i.e. participants are presented the two solutions and should assess which one they prefer the most. Therefore, after the ranking phase, participants are proposed the allocation stemming from the application of method A and of method B and asked whether they prefer one or the other.

The experiment was carried out in CESARE Lab, the experimental laboratory of LUISS University, Rome. Experimental subjects were undergraduate or postgraduate students from LUISS University of Rome. The experiment consisted of three phases:

- in Phase I participants faced a trust game (se Berg et al., 1995[54]) aimed at eliciting participants' trust in an unknown other before the more relevant division game;
- in Phase II participants were asked to express their preference between two different proposals of division stemming, respectively, from the two procedures defined for the CREA project. The computer then randomly selected one of the divisions and asked participants to accept or reject the outcome. If both components of the pair accepted the proposed allocation, it was implemented. If, instead, one of the two components of the pair rejected the proposed allocation, participants were sent to the court, where the judge would allocate half of the market

in Conflict Resolution: A Lab Experiment, in Algorithmic Conflict Resolution – Fair and Equitable Algorithms in Private Law (Romeo, F., Dall'Aglio, M. and Giacalone, M. eds), Giappichelli, 2019, 273-296.

[54] Berg, J., Dickhaut, J., and McCabe, K. (1995). Trust, reciprocity, and socialhistory.Games and economic behavior, 10(1), 122-142.

value of all the goods to each of them, and the litigants would have their payoff slightly reduced so to cover legal fees. Such division task was repeated for 10 rounds;

- in Phase III participants were randomly re-matched in pairs and face another round of the trust game to check whether litigations occurred in Phase II would have influenced trust. However, given the very low rejection rate of the proposed allocations, this effect could not be tested and therefore data from Phase III were not used in the analysis.

Experimental results show that the two procedures were equally preferred by participants (48,8% of the participants preferred the egalitarian versus 51, 2% of the Nash procedure) and these preferences remain constant throughout the repetitions of the game, which might be interpreted as a signal that stated choices are not case dependent.

Differences emerged when the gender was considered: women seem to have a higher preference for Nash solution while men prefer the Egalitarian one. This is consistent with the main experimental literature where female participants show to be more concerned than males about fairness.

Participants mostly accepted the division proposed by the computer, even when the preferred algorithm was not implemented (94.7% of divisions are accepted and therefore only 5.3% are rejected): only 5% of the cases were brought to court. They seem very unwilling to pay the cost of going to court and therefore accept the division even though this is not the preferred one. However, this could be interpreted also as their preference for allocations suggested by algorithms, as has been shown by other experimental evidences, especially in financial markets (see Alemanni et al, 2020)[55].

8. Results of the project

A close interaction between the analytical/experimental workgroup with the computational workgroup has been essential to com-

[55] Alemanni, B., Angelovski, A., di Cagno, D. T., Galliera, A., Linciano, N., Marazzi, F., and Soccorso, P. (2020). Do Investors Rely on Robots? Evidence from an Experimental Study. Evidence from an Experimental Study (September 21, 2020). CONSOB Fintech Series, (7).

plete the software implementation of the two procedures in such a short time. The software is now available for any user to experience at the project website http://www.crea-project.eu.

The mathematical results obtained in the context of the project. In Dall'Aglio (2019c)[56] and (2021)[57], the two procedures have been explained in high details, with the following improvements over the original description.

- It has been proved that, for any number of agents, a solution may be fair according to the Egalitarian or the Nash criterion and may deliver bundles of exactly equal monetary values only if the number of items that must be split among the agents is strictly higher than what it is necessary to achieve a fair solution without the additional requirement about the monetary value.

- Procedure 2 returns an allocation which is Egalitarian in the utilities. Differences in the monetary values of the bundles can be explained by the differences in the quality of the goods received. For instance, it is easy to compute the monetary value of the goods received by the three agents in example of the previous Section.

Agent	A	B	C
Monetary Value	204,614	209,197	218,189

Table 8 - Monetary value of the agents' bundles in the Example of Section 6

We may define an index that measures the quality, in terms of average standardized rating of the goods per percentage of monetary value received. We denote this measure as the rating-per-money (RM) index. When applied to the same three agents

Agent	A	B	C
RM Index	1.7948	1.5623	1.1208

[56] Dall'Aglio, M., (2019c): Fair Division of Goods with Market Values, 2019, arXiv.org, arXiv:1910.01615 [cs.GT].

[57] Dall'Aglio, M., (2021): Fair Division of Goods in the Shadow of Market Values, 2021, submitted.

Table 9 - Rating per Money (RM) index of the agents' bundles in the Example of Section 6

we notice that these indices rank opposite to the monetary values: the agent that received less in terms of monetary value has a higher RM index and, conversely, the agent that received more in terms of monetary value has the lowest RM index. This is no coincidence, and it is proved to happen in every situation. The following step can therefore be added in Procedure 2 immediately after Step III-i:

ii. The system computes the market value of the goods received by all agents and should explain possible differences in the market value of the received bundles by showing the differences in the average number of stars per fraction of good worth one unit of market value.

- When only two agents are involved, the solution for Procedure 2 can be given a simpler verbal and pictorial description that requires very little knowledge of the mathematical details. The description can guide the agents in assessing the correct rating for each good to be allocated.

9. Conclusions

The project has shown a fruitful collaboration among researchers from such diverse areas as law, mathematics, economics, and computer science. The team succeeded in delivering on time two fully functioning algorithms, which were developed starting from the data collected from real users in the legal field of work. The procedures were then fully validated through a lab experiment.

The project came to an end, 24 months after its start and having achieved all the expected goals. The whole process, however, would have benefited from further interactions among the researchers from such different areas. We outline some of these directions:

- In the project's final stages, a great effort was exerted in explaining the procedures to specialized audiences of law practitioners. This dissemination activity was limited to a handful of occasions, while a longer exposition of the project's findings and products would have helped agents in the legal field to became acquainted with the new tools offered by AI.

- The algorithms would have benefitted from further feedback from the legal agents on the validity of the model and on the elicitation of the parameters. For instance, the daily experience of judges and lawyers could help in setting up the range of bids in Procedure 1, or in defining the increase of utility that each additional rating level induces for a given good in Procedure 2.

- Experimental findings could be made even more relevant for the validity of the products by involving agents in the legal environment.

Notwithstanding the time limitation the project has set a new standard that should inspire similar projects on this specific or closely related issues.

The interaction between law and mathematics deserves further explorations: The mathematical models and algorithms provide powerful tools to make the apply the legal principles in their purest forms. In turn, law norms ignite new results in pure and applied mathematics. We believe this project has sown a relevant seed.